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Proof Theory for Minimal Quantum Logic II

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This paper provides proofs for the three main theorems of our previous paper which were enunciated without proofs there.

1. INTRODUCTION

In a previous paper (Nishimura, 1994) the three main proof-theoretical theorems were presented without proofs due to strict limitations of space. The objective of this paper is to give their relatively lengthy proofs in detail. We use freely the notation and terminology of our previous paper, assuming the reader to be familiar with it.

2. THE DUALITY THEOREM

Theorem 2.2 (The first duality theorem). If $\alpha \simeq \beta$, then $\alpha \simeq \beta''$.

Proof. It suffices to show the following four statements:

(I) If a sequent $\alpha, \Gamma \to \Delta$ is provable, then the sequent $\beta'', \Gamma \to \Delta$ is also provable.

(II) If a sequent $\Gamma \to \Delta, \alpha$ is provable, then the sequent $\Gamma \to \Delta, \beta''$ is also provable.

(I*) If a sequent $\alpha'', \Gamma \to \Delta$ is provable, then the sequent $\beta, \Gamma \to \Delta$ is also provable.

(II*) If a sequent $\Gamma \to \Delta, \alpha''$ is provable, then the sequent $\Gamma \to \Delta, \beta$ is also provable.

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It is easy to see that (I) and (II) follow at once from a simple application of the inference rules ($'' \to$) and ($\to ''$), respectively. (I*) and (II*) follow at once from the following, ostensibly more general statement.

(III) If $\alpha_1 \simeq \beta_1, \ldots, \alpha_n \simeq \beta_n$, $\alpha_{n+1} \simeq \beta_{n+1}, \ldots, \alpha_{n+m} \simeq \beta_{n+m}$ and a sequent $\alpha_1'', \ldots, \alpha_n''$, $\Gamma \to \Delta$, $\alpha_{n+1}'', \ldots, \alpha_{n+m}''$ has a proof P with $l(P) \leq k$, then the sequent $\beta_1, \ldots, \beta_n$, $\Gamma \to \Delta$, $\beta_{n+1}, \ldots, \beta_{n+m}$ is also provable.

We will prove (III) by induction on k. The proof is divided into cases according to which inference rule is used in the last step of P. To make the notation simpler, we proceed as if $n = 1$ and $m = 0$, leaving safely easy but due modifications to the reader. In dealing with the rules ($\wedge \to$), ($\to \vee$), ($\vee' \to$) and ($\to \wedge'$), each of which consists of two forms, we treat only one of them.

(a) The case that the sequent α_1'', $\Gamma \to \Delta$ is an axiom sequent: It must be that $\alpha_1'' \to \alpha_1''$. Since $\beta_1 \to \beta_1$ is an axiom sequent and $\alpha_1 \simeq \beta_1$ by assumption, the sequent $\beta_1 \to \alpha_1$ is provable, which implies that the sequent $\beta_1 \to \alpha_1''$ is also provable as follows:

$$\frac{\beta_1 \to \alpha_1}{\beta_1 \to \alpha_1''} \quad (\to '')$$

(b) The case that the last inference of the proof of the sequent α_1'', $\Gamma \to \Delta$ is (extension), ($\wedge \to$), ($\to \vee$), ($\wedge \to$), ($\to ''$), ($\vee' \to$), ($\to \wedge'$), ($\to \vee'$), or ($\vee \to'$): All the cases can be dealt with similarly, so here we deal only with the case in which the last inference of the proof is ($\to \wedge$) as follows:

$$\frac{\alpha_1'', \Gamma \to \beta \quad \alpha_1'', \Gamma \to \gamma}{\alpha_1'', \Gamma \to \beta \wedge \gamma} \quad (\to \wedge)$$

By the induction hypothesis the sequents $\beta_1, \Gamma \to \beta$ and $\beta_1, \Gamma \to \gamma$ are provable, which gives the desired result as follows:

$$\frac{\beta_1, \Gamma \to \beta \quad \beta_1, \Gamma \to \gamma}{\beta_1, \Gamma \to \beta \wedge \gamma} \quad (\to \wedge)$$

(c) The case that the last inference of the proof of α_1'', $\Gamma \to \Delta$ is ($'' \to$): Then the last inference is one of the following two forms.

$$\frac{\alpha_1, \Gamma \to \Delta}{\alpha_1'', \Gamma \to \Delta} \quad ('' \to) \qquad \frac{\alpha_1'', \beta, \Gamma_1 \to \Delta}{\alpha_1'', \beta'', \Gamma_1 \to \Delta} \quad ('' \to)$$

In the former case the sequent $\beta_1, \Gamma \to \Delta$ is provable for $\alpha_1 \simeq \beta_1$ and the sequent $\alpha_1, \Gamma \to \Delta$ is provable by assumption. In the latter case the sequent $\beta_1, \beta, \Gamma_1 \to \Delta$ is provable by the induction hypothesis, which implies that the sequent $\beta_1, \beta'', \Gamma_1 \to \Delta$ is provable as follows:

$$\frac{\beta_1, \beta, \Gamma_1 \to \Delta}{\beta_1, \beta'', \Gamma_1 \to \Delta} \quad ('' \to)$$

(d) The case that the last inference of the proof of the sequent $\alpha_1'', \Gamma \to \Delta$ is $(' \to)$: This case is divided into several subcases according to how the upper sequent of $(' \to)$ is obtained.

(d-1) The case that the upper sequent of $(' \to)$ is an axiom sequent: In this case the axiom sequent must be $\alpha_1' \to \alpha_1'$, so the proof that we must consider is as follows:

$$\frac{\alpha_1' \to \alpha_1'}{\alpha_1'', \alpha_1' \to} \quad (' \to)$$

Since the sequent $\alpha_1 \to \alpha_1$ is an axiom sequent and $\alpha_1 \simeq \beta_1$ by assumption, the sequent $\beta_1 \to \alpha_1$ is provable, which implies that the desired sequent $\beta_1, \alpha_1' \to$ is also provable as follows:

$$\frac{\beta_1 \to \alpha_1}{\alpha_1', \beta_1 \to} \quad (' \to)$$

(d-2) The case that the upper sequent of $(' \to)$ is obtained as the lower sequent of (extension), $(\wedge \to)$, $('' \to)$, or $(\vee ' \to)$: All these cases can be dealt with similarly, so here we consider only the case of $('' \to)$, in which the last two steps of the proof go as follows:

$$\cfrac{\cfrac{\beta, \Gamma_2 \to \alpha_1', \Gamma_1}{\beta'', \Gamma_2 \to \alpha_1', \Gamma_1} \quad ('' \to)}{\alpha_1'', \Gamma_1', \beta'', \Gamma_2 \to} \quad (' \to)$$

The sequent $\alpha_1'', \Gamma_1', \beta, \Gamma_2 \to$ has a shorter proof than the sequent $\alpha_1'', \Gamma_1', \beta'', \Gamma_2 \to$, as follows:

$$\frac{\beta, \Gamma_2 \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1', \beta, \Gamma_2 \to} \quad (' \to)$$

Therefore the sequent $\beta_1, \Gamma_1', \beta, \Gamma_2 \to$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma_1', \beta'', \Gamma_2 \to$ is also provable as follows:

$$\frac{\beta_1, \Gamma_1', \beta, \Gamma_2 \to}{\beta_1, \Gamma_1', \beta'', \Gamma_2 \to} \quad ('' \to)$$

(d-3) The case that the upper sequent of $(' \rightarrow)$ is obtained as the lower sequent of $(\rightarrow'')$: The last two steps of the proof that we must consider can be supposed to be one of the following two forms:

$$\cfrac{\cfrac{\Gamma_2 \rightarrow \beta, \Gamma_1}{\Gamma_2 \rightarrow \beta'', \Gamma_1} (\rightarrow'')}{\beta''', \Gamma_1', \Gamma_2 \rightarrow} (' \rightarrow)$$

$$\cfrac{\cfrac{\Gamma_2 \rightarrow \alpha_1', \beta, \Gamma_1}{\Gamma_2 \rightarrow \alpha_1', \beta'', \Gamma_1} (\rightarrow'')}{\alpha_1'', \beta''', \Gamma_1, \Gamma_2 \rightarrow} (' \rightarrow)$$

In the former case α_1 is supposed to be β'. Since the latter case can be dealt with in a similar manner to (d-2), here we deal with the former case, in which the sequent $\alpha_1, \Gamma_1', \Gamma_2 \rightarrow$ is provable with a shorter proof than that of the sequent $\alpha_1'', \Gamma_1', \Gamma_2 \rightarrow$ as follows:

$$\cfrac{\Gamma_2 \rightarrow \beta, \Gamma_1}{\beta', \Gamma_1', \Gamma_2 \rightarrow} (' \rightarrow)$$

Thus the desired sequent $\beta_1, \Gamma_1', \Gamma_2 \rightarrow$ is also provable by hypothesis.

(d-4) The case that the upper sequent of $(' \rightarrow)$ is obtained as the lower sequent of $(\rightarrow \vee)$: The last two steps of the proof go as follows:

$$\cfrac{\cfrac{\Gamma_2 \rightarrow \alpha_1', \beta, \Gamma_1}{\Gamma_2 \rightarrow \alpha_1', \beta \vee \gamma, \Gamma_1} (\rightarrow \vee)}{\alpha_1'', (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \rightarrow} (' \rightarrow)$$

The sequent $\alpha_1'', \beta', \Gamma_1', \Gamma_2 \rightarrow$ has a shorter proof than the sequent $\alpha_1'', (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \rightarrow$ as follows:

$$\cfrac{\Gamma_2 \rightarrow \alpha_1', \beta, \Gamma_1}{\alpha_1'', \beta', \Gamma_1', \Gamma_2 \rightarrow} (' \rightarrow)$$

Therefore the sequent $\beta_1, \beta', \Gamma_1', \Gamma_2 \rightarrow$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \rightarrow$ is provable as follows:

$$\cfrac{\beta_1, \beta', \Gamma_1', \Gamma_2 \rightarrow}{\beta_1, (\beta \vee \gamma)', \Gamma_1', \Gamma_2 \rightarrow} (\vee \rightarrow)$$

(d-5) The case that the upper sequent of $(' \rightarrow)$ is obtained as the lower sequent of $(\rightarrow \wedge')$; The last two steps of the proof are of one of the following two forms:

$$\cfrac{\cfrac{\Gamma_2 \to \alpha_1', \beta', \Gamma_1}{\Gamma_2 \to \alpha_1', (\beta \wedge \gamma)', \Gamma_1}\ (\to \wedge')}{\alpha_1'', (\beta \wedge \gamma)'', \Gamma_1', \Gamma_2 \to}\ ('\to)$$

$$\cfrac{\cfrac{\Gamma_2 \to \beta', \Gamma_1}{\Gamma_2 \to (\beta \wedge \gamma)', \Gamma_1}\ (\to \wedge')}{(\beta \wedge \gamma)'', \Gamma_1', \Gamma_2 \to}\ ('\to)$$

In the latter case α_1 is assumed to be $\beta \wedge \gamma$. Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. The sequent $\alpha_1'', \beta'', \Gamma_1', \Gamma_2 \to$ has a shorter proof than the sequent $\alpha_1'', (\beta \wedge \gamma)'', \Gamma_1', \Gamma_2 \to$ as follows:

$$\cfrac{\Gamma_2 \to \alpha_1', \beta', \Gamma_1}{\alpha_1'', \beta'', \Gamma_1', \Gamma_2 \to}\ ('\to)$$

This implies by the induction hypothesis that the sequent $\beta_1, \beta, \Gamma_1', \Gamma_2 \to$ is also provable. Thus the desired sequent $\beta_1, (\beta \wedge \gamma)'', \Gamma_1', \Gamma_2 \to$ is also provable, as follows:

$$\cfrac{\cfrac{\beta_1, \beta, \Gamma_1', \Gamma_2 \to}{\beta_1, \beta \wedge \gamma, \Gamma_1', \Gamma_2 \to}\ (\wedge \to)}{\beta_1, (\beta \wedge \gamma)'', \Gamma_1', \Gamma_2 \to}\ (''\to)$$

(d-6) The case that the upper sequent of $('\to)$ is obtained as the lower sequent of $(\vee \to)$: The last two steps of the proof that we must consider go as follows:

$$\cfrac{\cfrac{\beta \to \alpha_1', \Gamma_1 \quad \gamma \to \alpha_1', \Gamma_1}{\beta \vee \gamma \to \alpha_1', \Gamma_1}\ (\vee \to)}{\alpha_1'', \Gamma_1', \beta \vee \gamma \to}\ ('\to)$$

The sequents $\alpha_1'', \Gamma_1' \to \beta'$ and $\alpha_1'', \Gamma_1' \to \gamma'$ are provable with shorter proofs than that of $\alpha_1'', \Gamma_1', \beta \vee \gamma \to$ as follows

$$\cfrac{\beta \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \beta'}\ ('\to')$$

$$\cfrac{\gamma \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \gamma'}\ ('\to')$$

Therefore the sequents $\beta_1, \Gamma_1' \to \beta'$ and $\beta_1, \Gamma_1' \to \gamma'$ are provable by the induction hypothesis, which implies that the desired sequent $\beta \vee \gamma, \beta_1, \Gamma_1' \to$

is also provable as follows:

$$\frac{\beta_1, \Gamma_1' \to \beta' \quad \beta_1, \Gamma_1' \to \gamma'}{\beta \vee \gamma, \beta_1, \Gamma_1' \to} \ (\vee \to ')$$

(d-7) The case that the upper sequent of $(' \to)$ is obtained as the lower sequent of $(\wedge ' \to)$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\beta' \to \alpha_1', \Gamma_1 \quad \gamma' \to \alpha_1', \Gamma_1}{(\beta \wedge \gamma)' \to \alpha_1', \Gamma_1} \ (\wedge ' \to)}{\alpha_1'', \Gamma_1', (\beta \vee \gamma)' \to} \ (' \to)$$

The sequents $\alpha_1'', \Gamma_1' \to \beta''$ and $\alpha_1'', \Gamma_1' \to \gamma''$ are provable with shorter proofs than that of the sequent $\alpha_1'', \Gamma_1', (\beta \vee \gamma)' \to$ as follows:

$$\frac{\beta' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \beta''} \ (' \to ')$$

$$\frac{\gamma' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \gamma''} \ (' \to ')$$

Thus the sequents $\beta_1, \Gamma_1' \to \beta$ and $\beta_1, \Gamma_1' \to \gamma$ are provable by the induction hypothesis, which implies that the desired sequent $(\beta \wedge \gamma)', \beta_1, \Gamma_1' \to$ is also provable as follows:

$$\frac{\dfrac{\beta_1, \Gamma_1' \to \beta \quad \beta_1, \Gamma_1' \to \gamma}{\beta_1, \Gamma_1' \to \beta \wedge \gamma} \ (\to \wedge)}{(\beta \wedge \gamma)', \beta_1, \Gamma_1' \to} \ (' \to)$$

(d-8) The case that the upper sequent of $(' \to)$ is obtained as the lower sequent of $(\to \vee ')$: The last two steps of the proof that we must consider go as follows:

$$\frac{\dfrac{\Gamma_1 \to \beta' \quad \Gamma_1 \to \gamma'}{\Gamma_1 \to (\beta \vee \gamma)'} \ (\to \vee ')}{(\beta \vee \gamma)'', \Gamma_1 \to} \ (' \to)$$

Here α_1 is supposed to be $\beta \vee \gamma$. The sequent $\beta \vee \gamma, \Gamma_1 \to$ is provable as follows:

$$\frac{\Gamma_1 \to \beta' \quad \Gamma_1 \to \gamma'}{\beta \vee \gamma, \Gamma_1 \to} \ (\vee \to ')$$

Since $\beta_1 \simeq \alpha_1$ by assumption, the desired sequent $\beta_1, \Gamma_1 \to$ is also provable.

(d-9) The case that the upper sequent of $('\rightarrow)$ is obtained as the lower sequent of $('\rightarrow \wedge)$: The last two steps of the proof that we must consider go as follows:

$$\dfrac{\dfrac{\beta' \rightarrow \alpha_1', \Gamma_1 \quad \gamma' \rightarrow \alpha_1', \Gamma_1}{\rightarrow \beta \wedge \gamma, \alpha_1', \Gamma_1} \; ('\rightarrow \wedge)}{(\beta \wedge \gamma)', \alpha_1'', \Gamma_1' \rightarrow} \; ('\rightarrow)$$

The sequents $\alpha_1'', \Gamma_1' \rightarrow \beta$ and $\alpha_1'', \Gamma_1' \rightarrow \gamma''$ are provable with shorter proofs than that of the sequent $(\beta \wedge \gamma)', \alpha_1'', \Gamma_1' \rightarrow$ as follows:

$$\dfrac{\beta' \rightarrow \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \rightarrow \beta''} \; ('\rightarrow')$$

$$\dfrac{\gamma' \rightarrow \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \rightarrow \gamma''} \; ('\rightarrow')$$

Thus the sequents $\beta_1, \Gamma_1' \rightarrow \beta$ and $\beta_1, \Gamma_1' \rightarrow \gamma$ are provable by the induction hypothesis, which implies that the desired sequent $(\beta \wedge \gamma)', \beta_1, \Gamma_1' \rightarrow$ is also provable as follows:

$$\dfrac{\dfrac{\beta_1, \Gamma_1' \rightarrow \beta \quad \beta_1, \Gamma_1' \rightarrow \gamma}{\beta_1, \Gamma_1' \rightarrow \beta \wedge \gamma} \; (\rightarrow \wedge)}{(\beta \wedge \gamma)', \beta_1, \Gamma_1' \rightarrow} \; ('\rightarrow)$$

(d-10) The case that the upper sequent of $('\rightarrow)$ is obtained as the lower sequent of $('\rightarrow')$: The last two steps of the proof that we must consider go as follows:

$$\dfrac{\dfrac{\alpha_1, \Gamma_1 \rightarrow \Gamma_2}{\Gamma_2' \rightarrow \alpha_1', \Gamma_1'} \; ('\rightarrow')}{\alpha_1'', \Gamma_1'', \Gamma_2' \rightarrow} \; ('\rightarrow)$$

Since the sequent $\alpha_1, \Gamma_1 \rightarrow \Gamma_2$ is provable and $\alpha_1 \simeq \beta_1$ by assumption, $\beta_1, \Gamma_1 \rightarrow \Gamma_2$ is also provable, which implies that the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \rightarrow$ is provable, as follows:

$$\dfrac{\dfrac{\beta_1, \Gamma_1 \rightarrow \Gamma_2}{\beta_1, \Gamma_1, \Gamma_2' \rightarrow} \; ('\rightarrow)}{\beta_1, \Gamma_1'', \Gamma_2' \rightarrow} \; (''\rightarrow)$$

(d-11) The case that the upper sequent of $('\rightarrow)$ is obtained as the lower sequent of $(\rightarrow')$: We can proceed similarly to (d-10).

(e) The case that the last inference of the proof of the sequent $\alpha_1'', \Gamma \rightarrow \Delta$ is $('\rightarrow')$: This case is divided into several subcases according to how the upper sequent of $('\rightarrow')$ is obtained.

(e-1) The case that the upper sequent of $('\!\to')$ is an axiom sequent: The treatment of this case is similar to (d-1) and is safely left to the reader.

(e-2) The case that the upper sequent of $('\!\to')$ is obtained as the lower sequent of (extension): This case can safely be left to the reader.

(e-3) The case that the upper sequent of $('\!\to')$ is obtained as the lower sequent of $(\wedge \to)$: The last two steps of the proof that we must consider go as follows:

$$\dfrac{\dfrac{\alpha, \Delta_1 \to \alpha'_1, \Gamma_1}{\alpha \wedge \beta, \Delta_1 \to \alpha'_1, \Delta_1}\ (\wedge \to)}{\alpha''_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1}\ ('\!\to')$$

The sequent $\alpha''_1, \Gamma'_1 \to \alpha', \Delta'_1$ is provable with a shorter proof than that of $\alpha''_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1$ as follows:

$$\dfrac{\alpha, \Delta_1 \to \alpha'_1, \Delta_1}{\alpha''_1, \Gamma_1 \to \alpha', \Delta'_1}\ ('\!\to')$$

Thus the sequent $\beta_1, \Gamma'_1 \to \alpha', \Delta'_1$ is provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1$ is also provable as follows:

$$\dfrac{\beta_1, \Gamma'_1 \to \alpha', \Delta'_1}{\beta_1, \Gamma'_1 \to (\alpha \wedge \beta)', \Delta'_1}\ (\to \wedge')$$

(e-4) The case that the upper sequent of $('\!\to')$ is obtained as the lower sequent of $(\to \vee)$: The treatment is similar to (e-3) and is safely left to the reader.

(e-5) The case that the upper sequent of $('\!\to')$ is obtained as the lower sequent of $(\vee \to)$: The last two steps of the proof that we have to consider go as follows:

$$\dfrac{\dfrac{\alpha \to \alpha'_1, \Gamma_1 \quad \beta \to \alpha'_1, \Gamma_1}{\alpha \vee \beta \to \alpha'_1, \Gamma_1}\ (\vee \to)}{\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'}\ ('\!\to')$$

The sequents $\alpha''_1, \Gamma_1 \to \alpha'$ and $\alpha''_1, \Gamma'_1 \to \beta'$ are provable with shorter proofs than that of $\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'$ as follows:

$$\dfrac{\alpha \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \alpha'}\ ('\!\to') \qquad \dfrac{\beta \to \alpha'_1, \Gamma_1}{\alpha''_1, \Gamma'_1 \to \beta'}\ ('\!\to')$$

Thus the sequents $\beta_1, \Gamma'_1 \to \alpha'$ and $\beta_1, \Gamma'_1 \to \beta'$ are provable by the induction hypothesis, which implies that the desired sequent $\alpha''_1, \Gamma'_1 \to (\alpha \vee \beta)'$ is provable, as follows:

$$\frac{\beta_1, \Gamma_1' \to \alpha' \quad \beta_1, \Gamma_1' \to \beta'}{\beta_1, \Gamma_1' \to (\alpha \vee \beta)'} \; (\to \vee')$$

(e-6) The case that the upper sequent of $('\to')$ is the lower sequent of $(\to')$: The last two steps of the proof that we should consider can be supposed to be one of the following two forms:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Gamma_2}{\to \alpha_1', \Gamma_1', \Gamma_2} \; (\to')}{\alpha_1'', \Gamma_1'', \Gamma_2' \to} \; ('\to')$$

$$\cfrac{\cfrac{\Gamma_1 \to \alpha_1', \Gamma_2}{\to \alpha_1', \Gamma_1', \Gamma_2} \; (\to')}{\alpha_1'', \Gamma_1'', \Gamma_2' \to} \; ('\to')$$

In the former case the sequent $\alpha_1, \Gamma_1'', \Gamma_2' \to$ is provable as follows:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Gamma_2}{\alpha_1, \Gamma_1, \Gamma_2' \to} \; ('\to)}{\alpha_1, \Gamma_1'', \Gamma_2' \to} \; (''\to)$$

Since $\alpha_1 \simeq \beta_1$ by assumption, the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \to$ is also provable. As for the latter case, the sequent $\alpha_1'', \Gamma_2' \to \Gamma_1'$ is provable with a shorter proof than that of the sequent $\alpha_1'', \Gamma_1'', \Gamma_2' \to$, as follows:

$$\frac{\Gamma_1 \to \alpha_1', \Gamma_2}{\alpha_1'', \Gamma_2' \to \Gamma_1'} \; ('\to')$$

By the induction hypothesis the sequent $\beta_1, \Gamma_2' \to \Gamma_1'$ is also provable, which implies that the desired sequent $\beta_1, \Gamma_1'', \Gamma_2' \to$ is provable, as follows:

$$\frac{\beta_1, \Gamma_2' \to \Gamma_1'}{\beta_1, \Gamma_1'', \Gamma_2' \to} \; ('\to)$$

(e-7) The case that the upper sequent of $('\to')$ is obtained as the lower sequent of $(''\to)$ or $(\to'')$: The treatment is similar to (d-3) and is safely left to the reader.

(e-8) The case that the upper sequent of $('\to')$ is the lower sequent of another $('\to')$: The last two steps of the proof that we have to consider go as follows:

$$\cfrac{\cfrac{\alpha_1, \Gamma_1 \to \Delta_1}{\Delta_1' \to \alpha_1', \Gamma_1'} \; ('\to')}{\alpha_1'', \Gamma_1'' \to \Delta_1''} \; ('\to')$$

Since the sequent $\alpha_1, \Gamma_1 \to \Delta_1$ has a shorter proof than the sequent $\alpha_1'', \Gamma_1'' \to \Delta_1''$, the sequent $\beta_1, \Gamma_1 \to \Delta_1$ is also provable by the induction hypothesis, which implies that the desired sequent $\beta_1, \Gamma_1'' \to \Delta_1''$ is also provable, as follows

$$\cfrac{\cfrac{\beta_1, \Gamma_1 \to \Delta_1}{\beta_1, \Gamma_1'' \to \Delta_1} \; ('' \to)}{\beta_1, \Gamma_1'' \to \Delta_1''} \; (\to '')$$

(e-9) The case that the upper sequent of $(' \to ')$ is obtained as the lower sequent of $(\vee ' \to)$, $(\to \wedge ')$, $(\wedge ' \to)$, or $(\to \vee ')$: These four cases can be dealt with similarly, so here we deal only with the case of $(\to \vee ')$, in which the last two steps of the proof that we must consider go as follows:

$$\cfrac{\cfrac{\Delta_1 \to \alpha' \quad \Delta_1 \to \beta'}{\Delta_1 \to (\alpha \vee \beta)'} \; (\to \vee ')}{(\alpha \vee \beta)'' \to \Delta_1'} \; (' \to ')$$

Here α_1 is supposed to be $\alpha \vee \beta$. The sequents $\alpha'' \to \Delta_1'$ and $\beta'' \to \Delta_1'$ are provable with shorter proofs than that of $(\alpha \vee \beta)'' \to \Delta_1'$ as follows:

$$\cfrac{\Delta_1 \to \alpha'}{\alpha'' \to \Delta_1'} \; (' \to ') \qquad \cfrac{\Delta_1 \to \beta'}{\beta'' \to \Delta_1'} \; (' \to ')$$

Therefore the sequents $\alpha \to \Delta_1'$ and $\beta \to \Delta_1'$ are provable by the induction hypothesis, which implies that the sequent $\alpha \vee \beta \to \Delta_1'$ is also provable, as follows:

$$\cfrac{\alpha \to \Delta_1 \quad \beta \to \Delta_1}{\alpha \vee \beta \to \Delta_1} \; (\vee \to)$$

Since $\beta_1 \simeq \alpha_1 = \alpha \vee \beta$ by assumption, the desired sequent $\beta_1 \to \Delta_1$ is provable.

(e-10) The case that the upper sequent of $(' \to ')$ is obtained as the lower sequent of $(' \to \wedge)$: The last two steps of the proof that we must consider go as follows:

$$\cfrac{\cfrac{\alpha' \to \alpha_1', \Gamma_1 \quad \beta' \to \alpha_1', \Gamma_1}{\to \alpha_1', \Gamma_1, \alpha \wedge \beta} \; (' \to \wedge)}{\alpha_1'', \Gamma_1', (\alpha \wedge \beta)' \to} \; (' \to ')$$

The sequents $\Gamma_1', \alpha_1'' \to \alpha''$ and $\Gamma_1', \alpha_1'' \to \beta''$ are provable with shorter proofs than that of $\alpha_1'', \Gamma_1', (\alpha \wedge \beta)'' \to$ as follows:

$$\cfrac{\alpha' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \alpha''} \; (' \to ') \qquad \cfrac{\beta' \to \alpha_1', \Gamma_1}{\alpha_1'', \Gamma_1' \to \beta''} \; (' \to ')$$

By the induction hypothesis the sequents $\beta_1, \Gamma_1' \to \alpha$ and $\beta_1, \Gamma_1' \to \beta$ are provable, which implies that the desired sequent $\beta_1, \Gamma_1', (\alpha \wedge \beta)' \to$ is also provable, as follows:

$$\dfrac{\dfrac{\beta_1, \Gamma_1' \to \alpha \quad \beta_1, \Gamma_1' \to \beta}{\beta_1, \Gamma_1' \to \alpha \wedge \beta}\ (\to \wedge)}{\beta_1, \Gamma_1, (\alpha \wedge \beta)' \to}\ ('\to)$$

Theorem 2.4 (The second duality theorem). If $\alpha \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$, then $\alpha_1 \wedge \alpha_2 \simeq (\beta_1' \vee \beta_2')$ and $\alpha_1 \vee \alpha_2 \simeq (\beta_1' \wedge \beta_2')'$.

Proof. First we show the following four statements.

(I) If a sequent $\alpha_1 \wedge \alpha_2, \Gamma \to \Delta$ is provable, then the sequent $(\beta_1' \vee \beta_2')', \Gamma \to \Delta$ is also provable.

(II) If a sequent $\Gamma \to \Delta, \alpha_1 \wedge \alpha_2$ is provable, then the sequent $\Gamma \to \Delta, (\beta_1' \vee \beta_2')'$ is also provable.

(III) If a sequent $\alpha_1 \vee \alpha_2, \Gamma \to \Delta$ is provable, then the sequent $(\beta_1' \wedge \beta_2'), \Gamma \to \Delta$ is also provable.

(IV) If a sequent $\Gamma \to \Delta, \alpha_1 \vee \alpha_2$ is provable, then the sequent $\Gamma \to \Delta, (\beta_1' \wedge \beta_2')'$ is also provable.

Here we deal only with (II), leaving the remaining three statements to the reader. The proof is carried out by induction on the construction of a proof P of the sequent $\Gamma \to \Delta, \alpha_1 \wedge \alpha_2$. Here we deal only with the critical case in which the last inference is $(\to \wedge)$ as follows:

$$\dfrac{\Gamma \to \alpha_1 \quad \Gamma \to \alpha_2}{\Gamma \to \alpha_1 \wedge \alpha_2}\ (\to \wedge)$$

Since $\alpha_1 \simeq \beta_1$ and $\alpha_2 \simeq \beta_2$ by assumption, the sequents $\Gamma \to \beta_1$ and $\Gamma \to \beta_2$ are provable, which implies that the sequent $\Gamma'' \to (\beta_1' \vee \beta_2')'$ is provable, as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \to \beta_1}{\beta_1' \to \Gamma'}\ ('\to') \qquad \dfrac{\Gamma \to \beta_2}{\beta_2' \to \Gamma'}\ ('\to')}{\beta_1' \vee \beta_2' \to \Gamma'}\ (\vee \to)}{\Gamma'' \to (\beta_1' \vee \beta_2')'}\ ('\to')$$

Therefore the sequent $\Gamma \to (\beta_1' \vee \beta_2')'$ is provable by Theorem 2.2.

To establish the remaining half of the theorem smoothly, we introduce a useful notion weaker than provability equivalence. A wff β is said to *be provably dominated by* a wff α, in notation $\alpha \precsim \beta$, if we have that for any finite sets Γ and Δ of wffs:

(a) Whenever the sequent $\alpha, \Gamma \to \Delta$ is provable, the sequent $\beta, \Gamma \to \Delta$ is also provable.

(b) Whenever the sequent $\Gamma \to \Delta, \alpha$ is provable, the sequent $\Gamma \to \Delta, \beta$ is also provable.

We notice that what we have really proved in (I)–(IV) is that if $\gamma_1 \precsim \delta_1$ and $\gamma_2 \precsim \delta_2$, then $\gamma_1 \wedge \gamma_2 \precsim (\delta_1' \vee \delta_2')'$ and $\gamma_1 \vee \gamma_2 \precsim (\delta_1' \wedge \delta_2')'$. Similarly, what we have really proved in the proof of Theorem 2.1 is that if $\gamma_1 \precsim \delta_1$ and $\gamma_2 \precsim \delta_2$, then $\gamma_1' \precsim \delta_1'$, $\gamma_1 \wedge \gamma_2 \precsim \delta_1 \wedge \delta_2$ and $\gamma_1 \vee \gamma_2 \precsim \delta_1 \vee \delta_2$, while what we have really proved in the proof of Theorem 2.2 is that if $\alpha \precsim \beta$, then $\alpha'' \precsim \beta$. It is easy to see that two wffs α and β are provably equivalent iff each of them is provably dominated by the other. Thus, to conclude the proof of the theorem, it suffices to notice that

$$\alpha_1 \wedge \alpha_2 \precsim (\beta_1' \vee \beta_2')' \precsim (\alpha_1'' \wedge \alpha_2'')'' \precsim \alpha_1 \wedge \alpha_2$$

3. THE CUT-ELIMINATION THEOREM

Theorem 3.5 (The cut-elimination theorem). If sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ are provable with $\Delta_1 = \varnothing$ or $\Gamma_2 = \varnothing$, then the sequent $\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2$ is also provable. In other words, (cut) is permissible in GMQL.

Proof. Suppose that the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ have proofs P_1 and P_2, respectively. We prove the theorem by double induction principally on $g(\alpha)$ and secondarily on $l(P_1) + l(P_2)$. By Theorem 2.4 we can assume that there is no occurrence of the disjunction symbol $\vee$ in P_1 or P_2. As in the proof of Theorem 2.2, whenever we are forced to deal with the rules ($\wedge \to$) or ($\to \wedge'$), each of which consists of two forms, only one of them is treated. Our proof is divided into several cases according to which inference rule is used in the last step of P_1 or P_2 as follows:

(a) The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is an axiom sequent: There is nothing to prove.

(b) The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of (extension): Here we deal only with the case that the former sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of (extension), leaving the dual case to the reader. Then the last step of the proof P_1 is in one of the following two forms:

$$\frac{\Gamma_{11} \to \Delta_{11}, \alpha}{\Gamma_{11}, \Gamma_{12} \to \Delta_{11}, \Delta_{12}, \alpha} \quad \text{(extension)}$$

$$\frac{\Gamma_{11} \to \Delta_{11}}{\Gamma_{11}, \Gamma_{12} \to \Delta_{11}, \Delta_{12}, \alpha} \quad \text{(extension)}$$

In the former case the desired sequent $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2$ is provable by induction hypothesis as follows.

$$\cfrac{\cfrac{\Gamma_{11} \to \Delta_{11}, \alpha \quad \alpha, \Gamma_2 \to \Delta_2}{\Gamma_{11}, \Gamma_2 \to \Delta_{11}, \Delta_2} \text{(cut)}}{\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2} \text{(extension)}$$

In the latter case the desired sequent $\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2$ is obtained as follows.

$$\frac{\Gamma_{11} \to \Delta_{11}}{\Gamma_{11}, \Gamma_{12}, \Gamma_2 \to \Delta_{11}, \Delta_{12}, \Delta_2} \text{(extension)}$$

(c) The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of one of the inference rules $(''\!\to)$ and $(\wedge \to)$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of one of the inference rules $(\to'')$ and $(\to \wedge')$: Here we deal only with the case that the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to \wedge')$, leaving the remaining three cases to the reader. So the last step of P_2 is of the following two forms:

$$\frac{\alpha, \Gamma_2 \to \Sigma_2, \beta'}{\alpha, \Gamma_2 \to \Sigma_2, (\beta \vee \gamma)'} (\to \wedge')$$

The desired sequent $\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, (\beta \wedge \gamma)'$ is provable by induction hypothesis as follows:

$$\cfrac{\cfrac{\Gamma_1 \to \Delta_1, \alpha \quad \alpha, \Gamma_2 \to \Sigma_2, \beta'}{\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, \beta'} \text{(cut)}}{\Gamma_1, \Gamma_2 \to \Delta_1, \Sigma_2, (\beta \wedge \gamma)'} (\to \wedge')$$

(d) The case that either the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\wedge' \to)$ or the sequent $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to \wedge)$: Here we deal only with the former case, leaving a similar treatment of the latter case to the reader. So the last step of P_1 goes as follows:

$$\frac{\beta' \to \Delta_1, \alpha \quad \gamma' \to \Delta_1, \alpha}{(\beta \wedge \gamma)' \to \Delta_1, \alpha} (\wedge' \to)$$

If $\Gamma_2 = \varnothing$, then the desired sequent $(\beta \wedge \gamma)' \to \Delta_1, \Delta_2$ is provable by the induction hypothesis as follows:

$$\cfrac{\cfrac{\beta' \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\beta' \to \Delta_1, \Delta_2} \text{(cut)} \qquad \cfrac{\gamma' \to \Delta_1, \alpha \quad \alpha \to \Delta_2}{\gamma' \to \Delta_1, \Delta_2} \text{(cut)}}{(\beta \wedge \gamma)' \to \Delta_1, \Delta_2} (\wedge' \to)$$

Unless $\Gamma_2 = \varnothing$, the situation can be classified into cases according to which inference rule is used in the last step of P_2. If $\Gamma_2 \neq \varnothing$ and it is not the case that the last inference of P_2 is $(\rightarrow \wedge)$, the situation is subsumed under the cases that have been or will be dealt with. If $\Gamma_2 \neq \varnothing$ and the last inference of P_2 is $(\rightarrow \wedge)$, then surely $\Gamma_1 \neq \varnothing$, so that the situation can be handled dually to the case that $\Gamma_2 = \varnothing$.

(e) The case that either the sequent $\Gamma_1 \rightarrow \Delta_1, \alpha$ is obtained as the lower sequent of one of the inference rules $(\rightarrow'')$ and $(\rightarrow \wedge')$ or the sequent $\alpha, \Gamma_2 \rightarrow \Delta_2$ is obtained as the lower sequent of one of the inference rules $(''\rightarrow)$ and $(\wedge \rightarrow)$: Here we deal only with the case that the sequent $\Gamma_1 \rightarrow \Delta_1, \alpha$ is obtained as the lower sequent of $(\rightarrow \wedge')$, leaving the remaining three cases to the reader. So the last step of P_1 is in one of the following two forms:

$$\frac{\Gamma_1 \rightarrow \Sigma, \beta', \alpha}{\Gamma_1 \rightarrow \Sigma, (\beta \wedge \gamma)', \alpha} \ (\rightarrow \wedge')$$

$$\frac{\Gamma_1 \rightarrow \Delta_1, \beta'}{\Gamma_1 \rightarrow \Delta_1, (\beta \wedge \gamma)'} \ (\rightarrow \wedge')$$

In the latter case α is supposed to be $(\beta \wedge \gamma)'$. In the former case the (cut) at issue is an instance of (cut-1), so that $\Gamma_2 = \varnothing$, and the desired sequent $\Gamma_1 \rightarrow \Sigma, (\beta \wedge \gamma)', \Delta_2$ is provable by induction hypothesis, as follows:

$$\cfrac{\cfrac{\Gamma_1 \rightarrow \Sigma, \beta', \alpha \quad \alpha \rightarrow \Delta_2}{\Gamma_1 \rightarrow \Sigma, \beta'} \ (\text{cut})}{\Gamma_1 \rightarrow \Sigma, (\beta \wedge \gamma)'} \ (\rightarrow \wedge')$$

As for the latter case, the cut formula is $(\beta \wedge \gamma)'$, and the sequent $\beta', \Gamma_2 \rightarrow \Delta_2$ is provable by Corollary 3.4. Thus the desired sequent $\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2$ is provable by the induction hypothesis, as follows:

$$\frac{\Gamma_1 \rightarrow \Delta_1, \beta' \quad \beta', \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \ (\text{cut})$$

(f) The case that either the sequent $\Gamma_1 \rightarrow \Delta_1, \alpha$ is obtained as the lower sequent of $(\rightarrow \wedge)$ or the sequent $\alpha, \Gamma_2 \rightarrow \Delta_2$ is obtained as the lower sequent of $(\wedge' \rightarrow)$: Here we deal only with the latter case, leaving a similar treatment of the latter to the reader. So the last step of P_2 goes as follows:

$$\frac{\beta' \rightarrow \Delta_2 \quad \gamma' \rightarrow \Delta_2}{(\beta \wedge \gamma)' \rightarrow \Delta_2} \ (\wedge' \rightarrow)$$

Here α is supposed to be $(\beta \wedge \gamma)'$, and the (cut) at issue is an instance of (cut-1) with the cut formula $(\beta \wedge \gamma)'$. By Corollary 2.2 the sequent

$\Gamma_1 \to \Delta_1, \beta', \gamma'$ is provable, so that the desired sequent $\Gamma_1 \to \Delta_1, \Delta_2$ is also provable, as follows:

$$\dfrac{\dfrac{\Gamma_1 \to \Delta_1, \beta', \gamma' \quad \beta' \to \Delta_2}{\Gamma_1 \to \Delta_1, \Delta_2, \gamma'}\,(\text{cut}) \qquad \gamma' \to \Delta_2}{\Gamma_1 \to \Delta_1, \Delta_2}\,(\text{cut})$$

(g) The case that the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $('\!\to \wedge)$: The last step of P_1 is in one of the following two forms:

$$\dfrac{\beta' \to \Sigma, \alpha \quad \gamma' \to \Sigma, \alpha}{\to \Sigma, \beta \wedge \gamma, \alpha}\,('\!\to \wedge)$$

$$\dfrac{\beta' \to \Delta_1 \quad \gamma' \to \Delta_1}{\to \Delta_1, \beta \wedge \gamma}\,('\!\to \wedge)$$

In the latter case α is assumed to be $\beta \wedge \gamma$. First we deal with the former case, in which the (cut) at issue is (cut-1) so that $\Gamma_2 = \varnothing$. Then the desired sequent $\to \Sigma, \beta \wedge \gamma, \Delta_2$ is provable by the induction hypothesis, as follows:

$$\dfrac{\dfrac{\beta' \to \Sigma, \alpha \quad \alpha \to \Delta_2}{\beta' \to \Sigma, \Delta_2}\,(\text{cut}) \qquad \dfrac{\gamma' \to \Sigma, \alpha \quad \alpha \to \Delta_2}{\gamma' \to \Sigma, \Delta_2}\,(\text{cut})}{\to \Sigma, \beta \wedge \gamma, \Delta_2}\,('\!\to \wedge)$$

As for the latter case, suppose first that $\Delta_1 \neq \varnothing$, so that $\Gamma_2 = \varnothing$. Then the sequents $\Delta_1' \to \beta$ and $\Delta_1' \to \gamma$ are provable by Corollary 2.3, while the sequent $\beta, \gamma \to \Delta_2$ is provable by Theorem 3.1. Thus the sequent $\to \Delta_1'', \Delta_2$ is provable by the induction hypothesis, as follows:

$$\dfrac{\dfrac{\Delta_1' \to \gamma \quad \dfrac{\Delta_1' \to \beta \quad \beta, \gamma \to \Delta_2}{\gamma, \Delta_1' \to \Delta_2}\,(\text{cut})}{\Delta_1' \to \Delta_2}\,(\text{cut})}{\to \Delta_1'', \Delta_2}\,(\to ')$$

Thus the desired sequent $\to \Delta_1, \Delta_2$ is provable by Theorem 2.2. If $\Delta_1 = \varnothing$, then the sequents $\to \beta$ and $\to \gamma$ are provable by Corollary 2.3, while the sequent $\beta, \gamma, \Gamma_2 \to \Delta_2$ is provable by Theorem 3.1. Thus the desired sequent $\Gamma_2 \to \Delta_2$ is provable by the induction hypothesis, as follows:

$$\dfrac{\to \gamma \quad \dfrac{\to \beta \quad \beta, \gamma, \Gamma_2 \to \Delta_2}{\gamma, \Gamma_2 \to \Delta_2}\,(\text{cut})}{\Gamma_2 \to \Delta_2}\,(\text{cut})$$

(h) The case that one of the sequents $\Gamma_1 \to \Delta_1, \alpha$ and $\alpha, \Gamma_2 \to \Delta_2$ is obtained as the lower sequent of $(\to ')$: Here we deal only with the case that the sequent $\Gamma_1 \to \Delta_1, \alpha$ is obtained as the lower sequent of $(\to ')$, leaving the dual case to the reader. So the last step of the proof P_1 is in one of the following two forms:

$$\frac{\Delta_{12} \to \Delta_{11}, \alpha}{\to \Delta_{11}, \Delta'_{12}, \alpha} \ (\to')$$

$$\frac{\Delta_{12}, \beta \to \Delta_{11}}{\to \Delta_{11}, \Delta'_{12}, \beta'} \ (\to')$$

In the latter case α is supposed to be β'. First we deal with the former case. If $\Gamma_2 = \varnothing$, then the desired sequent $\to \Delta_{11}, \Delta'_{12}, \Delta_2$ is provable by the induction hypothesis, as follows:

$$\cfrac{\cfrac{\Delta_{12} \to \Delta_{11}, \alpha \quad \alpha \to \Delta_2}{\Delta_{12} \to \Delta_{11}, \Delta_2} \ (\text{cut})}{\to \Delta_{11}, \Delta'_{12}, \Delta_2} \ (\to')$$

If $\Gamma_2 \neq \varnothing$, then α is of the form γ' and the sequent $\Delta_{12} \to \Delta_{11}, \alpha$ is $\gamma \to \gamma'$. The sequents $\gamma \to$ and $\Delta'_2 \to \gamma$ are provable by Corollary 2.3, which implies that the sequent $\Delta'_2 \to$ is also provable by the induction hypothesis, as follows:

$$\frac{\Delta'_2 \to \gamma \quad \gamma \to}{\Delta'_2 \to} \ (\text{cut})$$

By Corollary 2.3 the sequent $\to \Delta_2$ is provable, which implies that the desired sequent $\to \Delta_1, \Delta_2$ is provable as follows:

$$\frac{\to \Delta_2}{\to \Delta_1, \Delta_2} \ (\text{extension})$$

Now we deal with the latter case. If $\Gamma_2 = \varnothing$, then the sequent $\Delta'_2 \to \beta$ is provable by Corollary 2.3, and the sequent $\to \Delta_{11}, \Delta'_{12}, \Delta''_2$ is also provable by the induction hypothesis as follows:

$$\cfrac{\cfrac{\Delta'_2 \to \beta \quad \Delta_{12}, \beta \to \Delta_{11}}{\Delta_{12}, \Delta'_2 \to \Delta_{11}} \ (\text{cut})}{\to \Delta_{11}, \Delta'_{12}, \Delta''_2} \ (\to')$$

Thus the desired sequent $\to \Delta_{11}, \Delta_{12}, \Delta_2$ is provable by Theorem 2.2. If $\Gamma_2 \neq \varnothing$, then the sequent $\Delta_{12}, \beta \to \Delta_{11}$ must be $\beta \to$ or $\beta \to \beta'$, the latter of which implies by Corollary 2.3 that the sequent $\beta \to$ is provable. Thus in any case the sequent $\beta \to$ is provable. Since the sequent $\Delta'_2 \to \Gamma'_2, \beta$ is provable by Corollary 2.3, the sequent $\Delta'_2 \to \Gamma'_2$ is provable by the induction hypothesis, as follows:

$$\frac{\Delta_2' \to \Gamma_2', \beta \quad \beta \to}{\Delta_2' \to \Gamma_2'} \text{ (cut)}$$

Therefore the sequent $\Gamma_2 \to \Delta_2$ is provable by Corollary 2.3, which implies that the desired sequent $\Gamma_2 \to \Delta_1, \Delta_2$ is provable as follows:

$$\frac{\Gamma_2 \to \Delta_2}{\Gamma_2 \to \Delta_1, \Delta_2} \text{ (extension)}$$

(i) The case that both the sequent $\Gamma_1 \to \Delta_1, \alpha$ and the sequent $\alpha, \Gamma_2 \to \Delta_2$ are obtained as the lower sequent of $('\to')$: The last steps of the proofs P_1 and P_2 go as follows:

$$\frac{\Sigma_1, \beta \to \Pi_1}{\Pi_1' \to \Sigma_1', \beta} \text{ } ('\to')$$

$$\frac{\Sigma_2 \to \beta, \Pi_2}{\beta', \Pi_2' \to \Sigma_2'} \text{ } ('\to')$$

In the above α is supposed to be β'. The desired sequent $\Pi_1, \Pi_2 \to \Sigma_1, \Sigma_2$ is provable by the induction hypothesis as follows:

$$\cfrac{\cfrac{\Sigma_2 \to \Pi_2, \beta \quad \beta, \Sigma_1 \to \Pi_1}{\Sigma_1, \Sigma_2 \to \Pi_1, \Pi_2} \text{ (cut)}}{\Pi_1', \Pi_2' \to \Sigma_1', \Sigma_2'} \text{ } ('\to')$$

NOTE ADDED IN PROOF

In our previous paper (Nishimura, 1994, p. 104) the inference rule

$$\frac{\alpha' \to \Delta \quad \beta' \to \Delta}{(\alpha \wedge \beta)' \to \Delta}$$

should have been named $(\wedge' \to)$, and the inference rule $(\to \vee')$ should have been

$$\frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\Gamma \to (\alpha \vee \beta)'}$$

REFERENCES

Nishimura, H. (1994). *International Journal of Theoretical Physics*, **33**, 103–113.